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# THE ANALYST.

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## NOTE ON A SPECIAL SYMMETRICAL DETERMINANT.

BY THOMAS MUIR, M. A., F. R. S. E., BEECHCROFT, SCOTLAND.

1. IN the Cambridge and Dublin Mathematical Journal, Vol. I, p. 286 (1846), a correspondent, signing himself H (1), gives the identity

$$\begin{aligned} & (a_1a_2 - b_1b_2 - c_1c_2)(b_1b_2 - c_1c_2 - a_1a_2)(c_1c_2 - a_1a_2 - b_1b_2) \\ & - (a_1a_2 - b_1b_2 - c_1c_2)(b_1c_2 + b_2c_1)^2 - (b_1b_2 - c_1c_2 - a_1a_2)(a_1c_2 + a_2c_1)^2 \\ & \quad - (c_1c_2 - a_1a_2 - b_1b_2)(a_1b_2 + a_2b_1)^2 \\ & + 2(b_1c_2 + b_2c_1)(a_1c_2 + a_2c_1)(a_1b_2 + a_2b_1) \\ & = (a_1^2 + b_1^2 + c_1^2)(a_1a_2 + b_1b_2 + c_1c_2)(a_2^2 + b_2^2 + c_2^2). \end{aligned}$$

No proof is added, and at first sight it might appear as if the verification of the identity would be a trifle laborious. The object of the present note is to give a proof interesting to some extent in itself and also as showing how a generalization of the identity may be effected.

2. The left hand member is expressible by a determinant, viz.,

$$\begin{vmatrix} a_1a_2 - b_1b_2 - c_1c_2 & a_1b_2 + a_2b_1 & a_1c_2 + a_2c_1 \\ a_1b_2 + a_2b_1 & b_1b_2 - c_1c_2 - a_1a_2 & b_1c_2 + b_2c_1 \\ a_1c_2 + a_2c_1 & b_1c_2 + b_2c_1 & c_1c_2 - a_1a_2 - b_1b_2 \end{vmatrix},$$

or, if we write  $S_3$  for  $a_1a_2 + b_1b_2 + c_1c_2$ , by

$$\begin{vmatrix} 2a_1a_2 - S_3 & a_1b_2 + a_2b_1 & a_1c_2 + a_2c_1 \\ a_1b_2 + a_2b_1 & 2b_1b_2 - S_3 & b_1c_2 + b_2c_1 \\ a_1c_2 + a_2c_1 & b_1c_2 + b_2c_1 & 2c_1c_2 - S_3 \end{vmatrix}.$$

Expanding this according to descending powers of  $S_3$  we have

$$\begin{aligned} & -S_3^3 + S_3^2(2a_1a_2 + 2b_1b_2 + 2c_1c_2) \\ & -S_3 \left\{ \begin{vmatrix} 2b_1b_2 & b_1c_2 + b_2c_1 \\ b_1c_2 + b_2c_1 & 2c_1c_2 \end{vmatrix} + \begin{vmatrix} 2a_1a_2 & a_1c_2 + a_2c_1 \\ a_1c_2 + a_2c_1 & 2c_1c_2 \end{vmatrix} \right\} \end{aligned}$$

$$+ \left| \begin{array}{cc} 2a_1a_2 & a_1b_2+a_2b_1 \\ a_1b_2+a_2b_1 & 2b_1b_2 \end{array} \right| \left\{ \right. \\ + \left| \begin{array}{ccc} 2a_1a_2 & a_1b_2+a_2b_1 & a_1c_2+a_2c_1 \\ a_1b_2+a_2b_1 & 2b_1b_2 & b_1c_2+b_2c_1 \\ a_1c_2+a_2c_1 & b_1c_2+b_2c_1 & 2c_1c_2 \end{array} \right| \\ \left. \right\}$$

where the term independent of  $S_3$

$$= \left| \begin{array}{ccc} a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \\ c_1 & c_2 & 0 \end{array} \right| \times \left| \begin{array}{ccc} a_2 & a_1 & 0 \\ b_2 & b_1 & 0 \\ c_2 & c_1 & 0 \end{array} \right| = 0$$

and the coefficient of the first power of  $S_3$

$$= \left| \begin{array}{cc} b_1 & c_1 \\ b_2 & c_2 \end{array} \right|^2 + \left| \begin{array}{cc} a_1 & c_1 \\ a_2 & c_2 \end{array} \right|^2 + \left| \begin{array}{cc} a_1 & b_1 \\ a_2 & b_2 \end{array} \right|^2 = \left| \begin{array}{ccc} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{array} \right|^2 \\ = \left| \begin{array}{cc} a_1^2 + b_1^2 + c_1^2 & a_1a_2+b_1b_2+c_1c_2 \\ a_1a_2+b_1b_2+c_1c_2 & a_2^2 + b_2^2 + c_2^2 \end{array} \right| \\ = (a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2) - S_3^2.$$

The original determinant is thus found

$$= -S_3^3 + S_3^2(2S_3) + S_3[(a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2) - S_3^2] \\ = S_3(a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2) \quad (1)$$

as was to be proved.

3. A glance at the steps of this proof suffices to suggest a direction in which the theorem may be extended. Writing  $S_4$  for  $a_1a_2 + b_1b_2 + c_1c_2 + d_1d_2$  we have

$$\left| \begin{array}{cccc} 2a_1a_2 - S_4 & a_1b_2 + a_2b_1 & a_1c_2 + a_2c_1 & a_1d_2 + a_2d_1 \\ a_1b_2 + a_2b_1 & 2b_1b_2 - S_4 & b_1c_2 + b_2c_1 & b_1d_2 + b_2d_1 \\ a_1c_2 + a_2c_1 & b_1c_2 + b_2c_1 & 2c_1c_2 - S_4 & c_1d_2 + c_2d_1 \\ a_1d_2 + a_2d_1 & b_1d_2 + b_2d_1 & c_1d_2 + c_2d_1 & 2d_1d_2 - S_4 \end{array} \right| \\ = S_4^4 - S_4^3(2a_1a_2 + 2b_1b_2 + 2c_1c_2 + 2d_1d_2) \\ + S_4^2(-|a_1b_2|^2 - |a_1c_2|^2 - |a_1d_2|^2 - |b_1c_2|^2 - |b_1d_2|^2 - |c_1d_2|^2) \\ - S_4(0 + 0 + 0 + 0) + 0 \left\{ \right. \\ = -S_4^4 + S_4^2[(a_1a_2 + b_1b_2 + c_1c_2 + d_1d_2)^2 - (a_1^2 + b_1^2 + c_1^2 + d_1^2)(a_2^2 + b_2^2 + c_2^2 + d_2^2)] \\ = -(a_1a_2 + b_1b_2 + c_1c_2 + d_1d_2)^2 (a_1^2 + b_1^2 + c_1^2 + d_1^2)(a_2^2 + b_2^2 + c_2^2 + d_2^2). \quad (2)$$

The transition from these two cases to the general theorem dealing with two sets of  $n$  letters can now be accomplished.

4. Putting  $d_1 = d_2 = 0$  in (2) and dividing both members by  $-(a_1a_2 + b_1b_2 + c_1c_2)$  we obtain (1); and similarly any case may be derived from that which follows it.

5. Putting  $a_1 = a_2 = a$ ,  $b_1 = b_2 = b$ ,  $c_1 = c_2 = c$ ,  $d_1 = d_2 = d$ , in (2) we have

$$\begin{vmatrix} a^2-b^2-c^2-d^2 & 2ab & 2ac & 2ad \\ 2ab & b^2-c^2-d^2-a^2 & 2bc & 2bd \\ 2ac & 2bc & c^2-d^2-a^2-b^2 & 2cd \\ 2ad & 2bd & 2cd & d^2-a^2-b^2-c^2 \end{vmatrix} = -(a^2+b^2+c^2+d^2)^4$$

and from this, by making  $d = 0$ , there results

$$\begin{vmatrix} a^2-b^2-c^2 & 2ab & 2ac \\ 2ab & b^2-c^2-a^2 & 2bc \\ 2ac & 2bc & c^2-a^2-b^2 \end{vmatrix} = (a^2+b^2+c^2)^3$$

and thence in the same way

$$\begin{vmatrix} a^2+b^2 & 2ab \\ 2ab & b^2-a^2 \end{vmatrix} = -(a^2+b^2)^2,$$

the identity well known in connection with Euc. I. 47, giving the sum of two squares as a square.

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## THE BITANGENTIAL.

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BY WILLIAM E. HEAL, MARION, INDIANA.

THE curve which passes through the points of contact of bitangents of a given curve is called the *bitangential* of that curve.

Such a curve may be determined by the method of problem 331, ANALYST. It is, however, desirable to obtain a curve of lower order, and for this purpose Salmon has given two methods in Higher Plane Curves.

Let us put

$$A = \begin{vmatrix} \frac{d^2u}{dy^2} & \frac{d^2u}{dz\,dy} \\ \frac{d^2u}{dy\,dz} & \frac{d^2u}{dz^2} \end{vmatrix}, \quad B = \begin{vmatrix} \frac{d^2u}{dx^2} & \frac{d^2u}{dz\,dx} \\ \frac{d^2u}{dx\,dz} & \frac{d^2u}{dz^2} \end{vmatrix}, \quad C = \begin{vmatrix} \frac{d^2u}{dx^2} & \frac{d^2u}{dy\,dx} \\ \frac{d^2u}{dx\,dy} & \frac{d^2u}{dy^2} \end{vmatrix},$$

$$D = \begin{vmatrix} \frac{d^2u}{dx\,dz} & \frac{d^2u}{dz\,dy} \\ \frac{d^2u}{dx^2} & \frac{d^2u}{dx\,dy} \end{vmatrix}, \quad E = \begin{vmatrix} \frac{d^2u}{dx\,dy} & \frac{d^2u}{dx\,dz} \\ \frac{d^2u}{dy^2} & \frac{d^2u}{dy\,dz} \end{vmatrix}, \quad F = \begin{vmatrix} \frac{d^2u}{dy\,dz} & \frac{d^2u}{dx\,dy} \\ \frac{d^2u}{dz^2} & \frac{d^2u}{dx\,dz} \end{vmatrix};$$